一般項が単項式×２の累乗である数列の和

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【2008 信州大】　和 $\sum_{k=1}^{n} k\,2^k$ を求めよ。

解答　$S_n=\sum_{k=1}^{n} k\,2^k$ とおくと，

$$S_n=1\cdot 2+2\cdot 2^2+3\cdot 2^3+\cdots\cdots+n2^n$$

$$2S_n=\qquad 1\cdot 2^2+2\cdot 2^3+\cdots\cdots+(n-1)\cdot 2^n+n2^{n+1}$$

辺々を引くと，

$$-S_n=2+2^2+2^3+\cdots\cdots+2^n-n2^{n+1}=\frac{2(2^n-1)}{2-1}-n2^{n+1}=(1-n)\cdot 2^{n+1}-2$$

よって，$S_n=(n-1)\cdot 2^{n+1}+2$　　すなわち　$\sum_{k=1}^{n} k\,2^k=(n-1)\cdot 2^{n+1}+2$　答

この問題から，一般項が単項式×２の累乗である数列の和を求めてみた。

(Ⅰ) $\sum_{k=1}^{n} k\cdot 2^k$　(Ⅱ) $\sum_{k=1}^{n} k^2\cdot 2^k$　(Ⅲ) $\sum_{k=1}^{n} k^3\cdot 2^k$　(Ⅳ) $\sum_{k=1}^{n} k^4\cdot 2^k$　(Ⅴ) $\sum_{k=1}^{n} k^5\cdot 2^k$

解答　$S_i=\sum_{k=1}^{n} k^i\cdot 2^k$　…①とおき，S_i と $2S_i$ の差をとる。

$$2S_i=\sum_{k=1}^{n} k^i\cdot 2^{k+1}=\sum_{k=1}^{n}(k-1)^i\cdot 2^k+n^i\cdot 2^{n+1}\quad\cdots②$$

①−②より，$-S_i=\sum_{k=1}^{n}\{k^i-(k-1)^i\}2^k-n^i\cdot 2^{n+1}$

$$\therefore \boxed{S_i=-\sum_{k=1}^{n}\{k^i-(k-1)^i\}2^k+n^i\cdot 2^{n+1}}\quad\cdots③$$

(Ⅰ) ③で，$i=0$ とおくと，$S_0=\sum_{k=1}^{n} 2^k=\frac{2(2^n-1)}{2-1}=2^{n+1}-2$

③で，$i=1$ とおくと，$S_1=-\sum_{k=1}^{n} 1\cdot 2^k+n\cdot 2^{n+1}=-S_0+n\cdot 2^{n+1}=-(2^{n+1}-2)+n\cdot 2^{n+1}$　（$\because$ S_0 を代入）

$$=(n-1)2^{n+1}+2\quad 答$$

(Ⅱ) ③で，$i=2$ とおくと，$S_2=-\sum_{k=1}^{n}(2k-1)\cdot 2^k+n^2\cdot 2^{n+1}=-2S_1+S_0+n^2\cdot 2^{n+1}$

$$=-2\{2^{n+1}(n-1)+2\}+(2^{n+1}-2)+n^2\cdot 2^{n+1}\quad(\because S_0,\ S_1 を代入)$$

$$=(n^2-2n+3)2^{n+1}-6\quad 答$$

(Ⅲ) ③で，$i=3$ とおくと，$S_3=-\sum_{k=1}^{n}(3k^2-3k+1)\cdot 2^k+n^3\cdot 2^{n+1}=-3S_2+3S_1-S_0+n^3\cdot 2^{n+1}$

S_0，S_1，S_2 を代入すると，$S_3=(n^3-3n^2+9n-13)2^{n+1}+26$　答

(Ⅳ) ③で，$i=4$ とおくと，$S_4=-\sum_{k=1}^{n}(4k^3-6k^2+4k-1)2^k+n^4\cdot 2^{n+1}$

$$=-4S_3+6S_2-4S_1+S_0+n^4\cdot 2^{n+1}$$

$S_0 \sim S_3$ を代入すると，$S_4=(n^4-4n^3+18n^2-52n+75)2^{n+1}-150$ 答

(V) ③で，$i=5$ とおくと，$S_5=-\sum_{k=1}^{n}(5k^4-10k^3+10k^2-5k+1)2^k+n^5\cdot 2^{n+1}$

$$=-5S_4+10S_3-10S_2+5S_1-S_0+n^5\cdot 2^{n+1}$$

$S_0 \sim S_4$ を代入すると，$S_5=(n^5-5n^4+30n^3-130n^2+375n-541)2^{n+1}+1082$ 答

参考 この方法を繰り返していくと，次式が得られる。

$S_6=\sum_{k=1}^{n}k^6\cdot 2^k=(n^6-6n^5+45n^4-260n^3+1125n^2-3246n+4683)2^{n+1}-9366$

$S_7=\sum_{k=1}^{n}k^7\cdot 2^k=(n^7-7n^6+63n^5-455n^4+2625n^3-11361n^2+32781n-47293)2^{n+1}+94586$

$S_8=\sum_{k=1}^{n}k^8\cdot 2^k=(n^8-8n^7+84n^6-728n^5+5250n^4-30296n^3+131124n^2-378344n+545835)2^{n+1}-1091670$

$S_9=\sum_{k=1}^{n}k^9\cdot 2^k=(n^9-9n^8+108n^7-1092n^6+9450n^5-68166n^4+393372n^3-1702548n^2+4912515n$

$-7087261)2^{n+1}+14174522$

$S_{10}=\sum_{k=1}^{n}k^{10}\cdot 2^k=(n^{10}-10n^9+135n^8-1560n^7+15750n^6-136332n^5+983430n^4-5675160n^3+24562575n^2$

$-70872610n+102247563)2^{n+1}-204495126$

別解

$f_i(x)=\sum_{k=1}^{n}k^i\cdot x^k=x+2^ix^2+3^ix^3+\cdots+n^ix^n$ とおくと，求める式は，$f_i(2)$（$i=1,\ 2,\ \cdots,\ 5$）である。

(I) $x \fallingdotseq 1$ のとき，$f_0(x)$ は等比数列の和であるから，$f_0(x)=x+x^2+x^3+\cdots+x^n=\dfrac{x(x^n-1)}{x-1}$

$$f_0'(x)=1+2x+3x^2+\cdots+nx^{n-1}=\frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}$$

$$xf_0'(x)=x+2x^2+3x^3+\cdots+nx^n=x\cdot\frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}=f_1(x)$$

$\therefore f_1(2)=(n-1)2^{n+1}+2$ 答

(II) 以下，(I)と同様に，微分して，x を掛け，2 を代入していく。

$$f_1'(x)=1+2^2x+3^2x^2+\cdots+n^2x^{n-1}=\frac{n^2x^{n+2}-(2n^2+2n-1)x^{n+1}+(n+1)^2x^n-x-1}{(x-1)^3}$$

$$xf_1'(x)=x+2^2x^2+3^2x^3+\cdots+n^2x^n=x\cdot\frac{n^2x^{n+2}-(2n^2+2n-1)x^{n+1}+(n+1)^2x^n-x-1}{(x-1)^3}=f_2(x)$$

$\therefore f_2(2)=(n^2-2n+3)2^{n+1}-6$ 答

(III)

$$f_2'(x)=\frac{n^3x^{n+3}-(3n^3+3n^2-3n+1)x^{n+2}+(3n^3+6n^2-4)x^{n+1}-(n+1)^3x^n+x^2+4x+1}{(x-1)^4}$$

$$xf_2'(x)=x\cdot\frac{n^3x^{n+3}-(3n^3+3n^2-3n+1)x^{n+2}+(3n^3+6n^2-4)x^{n+1}-(n+1)^3x^n+x^2+4x+1}{(x-1)^4}=f_3(x)$$

$\therefore f_3(2)=(n^3-3n^2+9n-13)2^{n+1}+26$ 答

(IV)

$f_3{}'(x)=\dfrac{1}{(x-1)^5}[n^4x^{n+4}-(4n^4+4n^3-6n^2+4n-1)x^{n+3}+(6n^4+12n^3-6n^2-12n+11)x^{n+2}-(4n^4+12n^3$

$+6n^2-12n-11)x^{n+1}+(n+1)^4x^n-x^3-11x^2-11x-1]$

$xf_3{}'(x)=\dfrac{x}{(x-1)^5}[n^4x^{n+4}-(4n^4+4n^3-6n^2+4n-1)x^{n+3}+(6n^4+12n^3-6n^2-12n+11)x^{n+2}-(4n^4+12n^3$

$+6n^2-12n-11)x^{n+1}+(n+1)^4x^n-x^3-11x^2-11x-1]=f_4(x)$

$\therefore f_4(2)=(n^4-4n^3+18n^2-52n+75)2^{n+1}-150$　答

(V)

$f_4{}'(x)=\dfrac{1}{(x-1)^6}[n^5x^{n+5}-(5n^5+5n^4-10n^3+10n^2-5n+1)x^{n+4}+2(5n^5+10n^4-10n^3+25n-13)x^{n+3}$

$-2(5n^5+15n^4-30n^2+33)x^{n+2}+(5n^5+20n^4+20n^3-20n^2-50n-26)x^{n+1}-(n+1)^5x^n+x^4+26x^3+66x^2$

$+26x+1]$

$xf_4{}'(x)=\dfrac{x}{(x-1)^6}[n^5x^{n+5}-(5n^5+5n^4-10n^3+10n^2-5n+1)x^{n+4}+2(5n^5+10n^4-10n^3+25n-13)x^{n+3}$

$-2(5n^5+15n^4-30n^2+33)x^{n+2}+(5n^5+20n^4+20n^3-20n^2-50n-26)x^{n+1}-(n+1)^5x^n+x^4+26x^3+66x^2$

$+26x+1]=f_5(x)$

$\therefore f_5(2)=(n^5-5n^4+30n^3-130n^2+375n-541)2^{n+1}+1082$　答

参考

$f_5{}'(x)=\dfrac{1}{(x-1)^7}[n^6x^{n+6}-(6n^6+6n^5-15n^4+20n^3-15n^2+6n-1)x^{n+5}+(15n^6+30n^5-45n^4-20n^3$

$+135n^2-150n+57)x^{n+4}-2(10n^6+30n^5-15n^4-80n^3+75n^2+120n-151)x^{n+3}+(15n^6+60n^5+30n^4$

$-160n^3-150n^2+240n+302)x^{n+2}-(6n^6+30n^5+45n^4-20n^3-135n^2-150n-57)x^{n+1}-x^5-57x^4-302x^3$

$-302x^2-57x-1]$

$xf_5{}'(x)=\dfrac{x}{(x-1)^7}[n^6x^{n+6}-(6n^6+6n^5-15n^4+20n^3-15n^2+6n-1)x^{n+5}+(15n^6+30n^5-45n^4-20n^3$

$+135n^2-150n+57)x^{n+4}-2(10n^6+30n^5-15n^4-80n^3+75n^2+120n-151)x^{n+3}+(15n^6+60n^5+30n^4$

$-160n^3-150n^2+240n+302)x^{n+2}-(6n^6+30n^5+45n^4-20n^3-135n^2-150n-57)x^{n+1}-x^5-57x^4-302x^3$

$-302x^2-57x-1]=f_6(x)$

$\therefore f_6(2)=(n^6-6n^5+45n^4-260n^3+1125n^2-3246n+4683)2^{n+1}-9366$　答

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