私の数学散歩道（10）　2012 .02. 04.提出

ピアーズ・フォスター　簡約積分表の公式の証明 (2)

～三角関数を含む式～

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前回に引き続き、題記の積分表から数学Ⅲの大学受験に耐える役立ちそうな分野の公式を証明しておくのも有用と考え、取り上げることにした。

今回は「三角関数を含む式」とし 266.から 389.の公式より適宜選んで証明をつけた。
部分積分の公式にこれらを組み合わせることで、実際入試等に出てくる三角関数を含む積分の計算に殆ど対応でき、また公式によっては積分の練習問題としても使えると思う。

一部に逆三角関数等 高校の範囲を超える公式もあるがご承知下さい。なお対数記号の中は正、分母 $\neq 0$ とし積分定数を省略、記載は「公式編」、「証明編」の順とした。

(公式編)

266. $$\int f(\sin x,\cos x,\tan x,\cot x,\sec x,\cos ecx)dx = f\left(\frac{2z}{1+z^2},\frac{1-z^2}{1+z^2},\frac{2z}{1-z^2},\frac{1-z^2}{2z},\frac{1+z^2}{1-z^2},\frac{1+z^2}{2z}\right)\frac{2dz}{1+z^2}$$ 但し $z=\tan\dfrac{x}{2}$

系 $$\int f(\sin x,\cos x,\tan x,\cot x,\sec x,\cos ecx) = f\left(\frac{z}{\sqrt{1+z^2}},\frac{1}{\sqrt{1+z^2}},z,\frac{1}{z},\sqrt{1+z^2},\frac{\sqrt{1+z^2}}{z}\right)\frac{dz}{1+z^2}$$ 但し $z=\tan x$

267. $\int \sin xdx = -\cos x$

268. $\int \sin^2 xdx = \frac{1}{2}x-\frac{1}{4}\sin 2x = \frac{1}{2}x-\frac{1}{2}\sin x\cos x$

269. $\int \sin^3 xdx = -\frac{1}{3}\cos x(\sin^2 x+2)$

270. $\int \sin^n xdx = -\frac{1}{n}\sin^{n-1}x\cos x+\frac{n-1}{n}\int \sin^{n-2}xdx$

271. $\int \cos xdx = \sin x$

272 $\int \cos^2 xdx = \frac{1}{2}x+\frac{1}{4}\sin 2x = \frac{1}{2}x+\frac{1}{2}\sin x\cos x$

273 $\int \cos^3 xdx = -\frac{1}{3}\sin x(\cos^2 x+2)$

274. $\int \cos^n xdx = \frac{1}{n}\sin x\cos^{n-1}x+\frac{n-1}{n}\int \cos^{n-2}xdx$

275. $\int \sin x\cos xdx = \frac{1}{2}\sin^2 x$

276. $\int \sin^2 x\cos^2 xdx = -\frac{1}{8}(\frac{1}{4}\sin 4x - x)$

277. $\int \sin x\cos^m xdx = -\frac{\cos^{m+1}x}{m+1}$

278. $\int \sin^m x\cos xdx = \frac{\sin^{m+1}x}{m+1}$

279. $\int \cos^m x \sin^n x dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \cos^{m-2} x \sin^n x dx$

280. $\int \cos^m x \sin^n x dx = -\frac{\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x dx$

284. $\int \frac{dx}{\sin x \cos x} = \log \tan x$

285. $\int \frac{dx}{\cos x \sin^2 x} = \log \tan(\frac{\pi}{4} + \frac{x}{2}) - \cos ecx$

289. $\int \tan x dx = -\log \cos x$

290. $\int \tan^2 x dx = \tan x - x$

291. $\int \tan^n x dx == \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$

292. $\int \cot x dx = \log \sin x$

293. $\int \cot^2 x dx = -\cot x - x$

295. $\int \sec x dx = \log \tan(\frac{\pi}{4} + \frac{x}{2}) = \frac{1}{2} \log \frac{1+\sin x}{1-\sin x} = \log(\sec x + \tan x)$

$= \log \frac{\cos x}{1-\sin x} = \log \frac{1+\sin x}{\cos x} = -\log(\sec x - \tan x)$

296. $\int \sec^2 x dx = \tan x$

298. $\int \cos ecx dx = \log \tan \frac{x}{2} = \frac{1}{2} \log \frac{1-\cos x}{1+\cos x} = \log(\cos ecx - \cot x) = \log \frac{\sin x}{1+\cos x}$

$= \log \frac{1-\cos x}{\sin x} = -\log(\cos ecx + \cot x)$

299. $\int \cos ec^2 x dx = -\cot x$

301. $\int \frac{dx}{a+b\cos x} = \frac{1}{c(b-a)} (\int \frac{dz}{z+c} - \int \frac{dz}{z-c})$ （$z = \tan \frac{x}{2}$、 $c^2 = \frac{b+a}{b-a}$）

302. $\int \frac{dx}{a \pm b \sin x} = \int \frac{2dz}{a \pm 2bz + az^2}$ （$z = \tan \frac{x}{2}$）

303. $\int \frac{dx}{1+\sin x} = -\tan(\frac{\pi}{4} - \frac{x}{2})$

304. $\int \frac{dx}{1-\sin x} = \cot(\frac{\pi}{4} - \frac{x}{2}) = \tan(\frac{\pi}{4} + \frac{x}{2})$

305 $\int \frac{dx}{1+\cos x} = \tan \frac{x}{2} = \cos ecx - \cot x$

306. $\int \frac{dx}{1-\cos x} = -\cot \frac{x}{2} = -\cot x - \cos ecx$

307. $\int \frac{dx}{a+b\sin x}$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{a\tan(x/2)+b}{\sqrt{a^2-b^2}} \text{ あるいは } = \frac{1}{\sqrt{b^2-a^2}}\log\frac{a\tan(x/2)+b-\sqrt{b^2-a^2}}{a\tan(x/2)+b+\sqrt{b^2-a^2}} \text{ あるいは}$$

$$= \frac{-2}{\sqrt{b^2-a^2}}\tanh^{-1}\frac{a\tan(x/2)+b}{\sqrt{b^2-a^2}} \text{ あるいは } = \frac{-2}{\sqrt{b^2-a^2}}\coth^{-1}\frac{a\tan(x/2)+b}{\sqrt{b^2-a^2}} \qquad (|x|<\pi)$$

308. $\int \frac{dx}{a+b\sin x} = \frac{1}{b\cos\alpha}\log\frac{\sin\frac{x+\alpha}{2}}{\cos\frac{x-\alpha}{2}} \qquad a=b\sin\alpha \quad \sqrt{b^2-a^2}=b\cos\alpha \qquad (|x|<\pi)$

309. $\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{\sqrt{a^2-b^2}\tan(x/2)}{a+b}$

$$\text{あるいは} \quad = \frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b^2-a^2}\tan(x/2)+a+b}{\sqrt{b^2-a^2}\tan(x/2)-a-b} \qquad (|x|<\pi)$$

310. $\int \frac{dx}{a+b\tan x} = \frac{1}{a^2+b^2}\left[b\log(a\cos x+b\sin x)+ax\right]$

311. $\int \frac{dx}{\sin x+\cos x} = \frac{1}{\sqrt{2}}\log\tan(\frac{x}{2}+\frac{\pi}{8})$

312. $\int \frac{\sin x}{a+b\cos x}dx = -\frac{1}{b}\log(a+b\cos x)$

313. $\int \frac{(a'+b'\cos x)dx}{a+b\cos x} = \frac{b'}{b}x+\frac{a'b-ab'}{b}\int\frac{dx}{a+b\cos x}$

323. $\int \frac{dx}{a+b\sin^2 x} = \frac{1}{\sqrt{a^2+ab}}\tan^{-1}\frac{\sqrt{a^2+ab}\tan x}{a}$

$$\text{あるいは} \quad = \frac{1}{2\sqrt{-a^2-ab)}}\log\frac{\sqrt{-a^2-ab}\tan x+a}{\sqrt{-a^2-ab}\tan x-a} \qquad (|x|<\frac{\pi}{2})$$

324. $\int \frac{dx}{a+b\cos^2 x} = \frac{1}{\sqrt{a^2+ab}}\tan^{-1}\frac{\sqrt{a^2+ab}\tan x}{a+b}$

$$\text{あるいは} \quad = \frac{1}{2\sqrt{-a^2-ab)}}\log\frac{\sqrt{-a^2-ab}\tan x+a+b}{\sqrt{-a^2-ab}\tan x-a-b} \qquad (|x|<\frac{\pi}{2})$$

325. $\int \frac{dx}{a\cos^2 x+b\cos^2 x} = \frac{1}{\sqrt{ab}}\tan^{-1}\frac{\sqrt{ab}\tan x}{a} \text{ あるいは} = \frac{1}{2\sqrt{-ab}}\log\frac{\sqrt{-ab}\tan x+a}{\sqrt{-ab}\tan x-a} \quad (|x|<\frac{\pi}{2})$

326. $\int \frac{\sin x \cos x dx}{a\cos^2 x + b\cos^2 x} = \frac{1}{2(b-a)} \log(a\cos^2 x + b\cos^2 x)$

328. $\int \frac{dx}{a + b\cos x + c\sin x} = \frac{2}{\sqrt{a^2 - b^2 - c^2}} \tan^{-1} \frac{(a-b)\tan(x/2) + c}{\sqrt{a^2 - b^2 - c^2}}$

あるいは $= \frac{1}{\sqrt{b^2 + c^2 - a^2}} \log \frac{(a-b)\tan(x/2) + c - \sqrt{b^2 + c^2 - a^2}}{(a-b)\tan(x/2) + c + \sqrt{b^2 + c^2 - a^2}}$ $\quad (|x| < \pi)$

329. $\int \frac{dx}{a(1+\cos x) + c\sin x} = \frac{1}{c} \log(a + c\tan\frac{x}{2})$

331. $\int \frac{(x + \sin x)dx}{1 + \cos x} = x\tan\frac{x}{2}$

338. $\int \frac{xdx}{1 + \sin x} = -x\tan(\frac{\pi}{4} - \frac{x}{2}) + 2\log\cos(\frac{\pi}{4} - \frac{x}{2})$

339. $\int \frac{xdx}{1 - \sin x} = \ x\cot(\frac{\pi}{4} - \frac{x}{2}) + 2\log\sin(\frac{\pi}{4} - \frac{x}{2})$

340. $\int \frac{xdx}{1 + \cos x} = x\tan\frac{x}{2} + 2\log\cos\frac{x}{2}$

341. $\int \frac{xdx}{1 - \cos x} = -x\cot\frac{x}{2} + 2\log\sin\frac{x}{2}$

343. $\int \frac{dx}{a + b\tan^2 x} = \frac{1}{a-b}\{\ x - \sqrt{\frac{b}{a}}\tan^{-1}(\sqrt{\frac{b}{a}}\tan x)\}$

344. $\int \frac{\tan x dx}{a + b\tan x} = \frac{1}{a^2 + b^2}\{bx - a\log(a + b\tan x) + a\log\sec x\}$

345. $\int x\sin x dx = \sin x - x\cos x$

346. $\int x^2 \sin x dx = 2x\sin x - (x^2 - 2)\cos x$

347 $\int x^3 \sin x dx = (3x^2 - 6)\sin x - (x^3 - 6x)\cos x$

348. $\int x^m \sin x dx = -x^m \cos x + m\int x^{m-1} \cos x dx$

353. $\int \frac{\sin x dx}{x^m} = -\frac{1}{m-1} \cdot \frac{\sin x}{x^{m-1}} + \frac{1}{m-1}\int \frac{\cos x}{x^{m-1}} dx$

355. $\int \frac{xdx}{\sin^2 x} = -x\cot x + \log\sin x$

356. $\int \frac{xdx}{\cos^2 x} = x\tan x + \log\cos x$

364. $\int \sin mx \sin nx dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}$

365 $\int \sin mx \cos nx dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)}$

366 $\int \cos mx \cos nx dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)}$

369. $\int \sin(mx + a)\sin(nx + b)dx \ = -\frac{\sin\{(m+n)x + a + b]}{2(m+n)} + \frac{\sin\{(m-n)x + a - b\}}{2(m-n)}$

370. $\int \cos(mx+a)\cos(nx+b)dx = \dfrac{\sin\{(m+n)x+a+b]}{2(m+n)} + \dfrac{\sin\{(m-n)x+a-b\}}{2(m-n)}$

371. $\int \sin(mx+a)\cos(nx+b)dx = -\dfrac{\cos\{(m+n)x+a+b]}{2(m+n)} - \dfrac{\cos\{(m-n)x+a-b\}}{2(m-n)}$

372 $\int \sin^2 mxdx = \dfrac{1}{2m}(mx - \sin mx \cos mx)$

373. $\int \cos^2 mxdx = \dfrac{1}{2m}(mx + \sin mx \cos mx)$

374. $\int \sin mx \cos mxdx = -\dfrac{1}{4m}\cos 2mx$

375. $\int \sin nx \sin^m xdx = \dfrac{1}{m+n}\left\{-\cos nx \sin^m x + m\int \cos(n-1)x \cdot \sin^{m-1} xdx\right\}$

378. $\int \cos nx \cos^m xdx = \dfrac{1}{m+n}\left\{\sin nx \cos^m x + m\int \cos(n-1)x \cdot \cos^{m-1} xdx\right\}$

386. $\int \sin ax \sin bx \sin cxdx$

$$= -\frac{1}{4}\left[\frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c}\right]$$

387. $\int \cos ax \cos bx \cos cxdx$

$$= \frac{1}{4}\left[\frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(c+a-b)x}{c+a-b} + \frac{\sin(a+b-c)x}{a+b-c}\right]$$

388. $\int \sin ax \cos bx \cos cxdx$

$$= -\frac{1}{4}\left[\frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(c+a-b)x}{c+a-b}\right]$$

389. $\int \cos ax \sin bx \sin cxdx$

$$= \frac{1}{4}\left[\frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a-b+c)x}{a-b+c} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a}\right]$$

以　上

(参照文献)

ピアース・フォスター　簡約積分表（理工学海外名著シリーズ 6）　第 4 版

「三角関数を含む式」　　ブレイン図書出版(株)

（証明編）

266. $\int f(\sin x, \cos x, \tan x, \cot x, \sec x, \cos ecx)dx = f(\frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2}, \frac{2z}{1-z^2},$

$\frac{1-z^2}{2z}, \frac{1+z^2}{1-z^2}, \frac{1+z^2}{2z})\frac{2dz}{1+z^2}$　但し　$z = \tan(x/2)$　　$dz = \frac{dx}{2\cos^2(x/2)}$　より　$dx = \frac{2dz}{1+z^2}$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin(x/2)\cos(x/2)}{\sin^2(x/2)+\cos^2(x/2)} = \frac{2\tan(x/2)}{1+\tan^2(x/2)}$　　同様にして

$\cos x = \frac{\cos^2(x/2)-\sin^2(x/2)}{\sin^2(x/2)+\cos^2(x/2)} = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$　　倍角の公式から　$\tan x = \frac{2\tan(x/2)}{1-\tan^2(x/2)}$

注）$\cot x, \sec x, \cos ecx$　は各々　$\sin x, \cos x, \tan x$ の逆数より公式が成立する。系も同じ。

系　$\int f(\sin x, \cos x, \tan x, \cot x, \sec x, \cos ecx) = f(\frac{z}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+z^2}}, z,$

$\frac{1}{z}, \sqrt{1+z^2}, \frac{\sqrt{1+z^2}}{z})\frac{dz}{1+z^2}$　但し　$z = \tan x$　　$dz = \frac{dx}{\cos^2 x}$ より　$dx = \frac{dz}{1+z^2}$

$\sin x = \tan x\cos x = \frac{\tan x}{\frac{1}{\cos x}} = \frac{\tan x}{\sqrt{1+\tan^2 x}}$　　$\cos x = \frac{1}{\frac{1}{\cos x}} = \frac{1}{\sqrt{1+\tan^2 x}}$

267. $\int \sin xdx = -\cos x$

268. $\int \sin^2 xdx = \frac{1}{2}\int(1-\cos 2x)dx = \frac{1}{2}x - \frac{1}{4}\sin 2x = \frac{1}{2}x - \frac{1}{2}\sin x\cos x$

269. $\int \sin^3 xdx = \int(-\cos x)'\sin^2 xdx = -\cos x\sin^2 x + 2\int \sin x\cos^2 xdx = -\cos x\sin^2 x$

$-\frac{2}{3}\int(\cos^3 x)'dx = -\frac{1}{3}\cos x(2\cos^2 x + 3\sin^2 x) = -\frac{1}{3}\cos x(\sin^2 x + 2)$

270. $\int \sin^n xdx = \int(-\cos x)'\sin^{n-1} xdx = -\cos x\sin^{n-1} x + (n-1)\int \cos^2 x\sin^{n-2} xdx$

$= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} xdx - (n-1)\int \sin^n xdx$

同類項で整理し $\{(n-1)+1\}\int \sin^n xdx = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} xdx$

従って　$\int \sin^n xdx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} xdx$

271. $\int \cos xdx = \sin x$

272. $\int \cos^2 x dx = \frac{1}{2}(1+\cos 2x)dx = \frac{1}{2}x + \frac{1}{4}\sin 2x = \frac{1}{2}x + \frac{1}{2}\sin x \cos x$

273. $\int \cos^3 x dx = \frac{1}{4}\int (3\cos x + \cos 3x)dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x = \sin x - \frac{1}{3}\sin^3 x$

$$= \frac{1}{3}\sin x(3-\sin^2 x) = -\frac{1}{3}\sin x(\cos^2 x + 2)$$

274. $\int \cos^n x dx = \int (\sin x)' \cos^{n-1} x dx = \sin x \cos^{n-1} x + (n-1)\int \sin^2 x \cos^{n-2} x dx$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$$

これから　　$\int \cos^n x dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x dx$

275. $\int \sin x \cos x dx = \int (\sin x)' \sin x dx = \sin^2 x - \int \sin x \cos x dx$ より　与式$= \frac{1}{2}\sin^2 x$

276. $\int \sin^2 x \cos^2 x dx = \int \left\{ \frac{1-\cos 2x}{2}(1 - \frac{1-\cos 2x}{2}) \right\} dx = \frac{1}{2}\int (1-\cos 2x)dx$

$$-\frac{1}{4}\int (1-\cos 2x)^2 dx = \frac{1}{2}x - \frac{1}{4}\sin 2x - \frac{1}{4}\int (1 - 2\cos 2x + \frac{1+\cos 4x}{2})dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x - \frac{1}{4}(x - \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x) = -\frac{1}{8}(\frac{1}{4}\sin 4x - x)$$

277. $\int \sin x \cos^m x dx = -\int t^m dt = -\frac{t^{m+1}}{m+1} = -\frac{\cos^{m+1} x}{m+1}$　（$\cos x = t$ から　$\sin x dx = -dt$）

278. $\int \sin^m x \cos x dx = \int t^m dt = \frac{t^{m+1}}{m+1} = \frac{\sin^{m+1} x}{m+1}$　（$\sin x = t$ から　$\cos x dx = dt$）

279. $\int \cos^m x \sin^n x dx = I(m,n)$　として

$$I(m,n) = \int \left(\frac{\sin^{n+1} x}{n+1} \right)' \cdot \frac{\cos^m x}{\cos x} dx = \frac{\sin^{n+1} x \cos^{m-1} x}{n+1} + \frac{m-1}{n+1}\int \sin^n x \sin^2 x \cos^{m-2} x dx$$

$$= \frac{\sin^{n+1} x \cos^{m-1} x}{n+1} + \frac{m-1}{n+1}\int \sin^n x(1-\cos^2 x)\cos^{m-2} x dx$$

$$= \frac{\sin^{n+1} x \cos^{m-1} x}{n+1} + \frac{m-1}{n+1} I(m-2,n) - \frac{m-1}{n+1} I(m,n)$$

これから　$I(m,n) = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n} I(m-2,n)$　で成立する。

280. $\int \cos^m x \sin^n x dx = I(m,n)$　として

$$I(m,n)=-\int\left(\frac{\cos^{m+1}x}{m+1}\right)'\cdot\frac{\sin^n x}{\sin x}dx=-\frac{\cos^{m+1}x\sin^{n-1}x}{m+1}+\frac{n-1}{m+1}\int\cos^m x\cos^2 x\sin^{n-2}xdx$$

279.と同様に　$I(m,n)=-\dfrac{\cos^{m+1}x\sin^{n-1}x}{m+n}+\dfrac{n-1}{m+n}I(m,n-2)$　で成立する。

284. $\displaystyle\int\frac{dx}{\sin x\cos x}=\int\frac{1+z^2}{z(1-z^2)}dz=\int(\frac{1}{z}+\frac{2z}{1-z^2})dz=\log 2z-\log(1-z^2)-\log 2$

$$=\log\frac{2z}{1-z^2}-\log 2=\log\frac{2\tan(x/2)}{1-\tan^2(x/2)}-\log 2=\log\tan x-\log 2\qquad(z=\tan\frac{x}{2})$$

定数　$-\log 2$　を除き　$\displaystyle\int\frac{dx}{\sin x\cos x}dx=\log\tan x$

285. $\displaystyle\int\frac{dx}{\cos x\sin^2 x}=\frac{1}{2}\int\frac{(1+z^2)^2}{z^2(1-z^2)}dz=-\frac{1}{2}\int\left\{1+\frac{3z^2+1}{z^2(z^2-1)}\right\}dz=-\frac{1}{2}z-\frac{1}{2}\int(\frac{4}{z^2-1}-\frac{1}{z^2})dz$

（$z=\tan\dfrac{x}{2}$、公式 266.参照。以降、置換えの式のみ表示。）　$\displaystyle=-\frac{1}{2}z+\int(\frac{1}{1-z}+\frac{1}{1+z})dz-\frac{1}{2z}$

$$=-\frac{1}{2}(\tan\frac{x}{2}+\frac{1}{\tan(x/2)})+\log\frac{1+\tan(x/2)}{1-\tan(x/2)}=-\frac{\sin^2(x/2)+\cos^2(x/2)}{2\sin(x/2)\cos(x/2)}+\log\frac{1+\tan(x/2)}{1-\tan(x/2)}$$

$$=-\frac{1}{\sin x}+\log\frac{(1/\sqrt{2})\{cos(x/2)+\sin(x/2)|}{(1/\sqrt{2})\{cos(x/2)-\sin(x/2)\}}=\log\tan(\frac{\pi}{4}+\frac{x}{2})-\cos ecx$$

289. $\displaystyle\int\tan xdx=\int\frac{\sin x}{\cos x}dx=-\int\frac{(\cos x)'}{\cos x}dx=-\log\cos x$

290. $\displaystyle\int\tan^2 xdx=\int\frac{1-\cos^2 x}{\cos^2 x}dx=\int\frac{dx}{\cos^2 x}-\int dx=\tan x-x$

291. $\displaystyle\int\tan^n xdx=\int\tan^{n-2}x\tan^2 x=\int\tan^{n-2}x(\sec^2 x-1)dx$

$$=\int(\tan x)'\tan^{n-2}xdx-\int\tan^{n-2}xdx=\frac{\tan^{n-1}x}{n-1}-\int\tan^{n-2}xdx$$

292. $\displaystyle\int\cot xdx=\int\frac{(\sin x)'}{\sin x}dx=\log\sin x$

293. $\displaystyle\int\cot^2 xdx=\int\frac{1-\sin^2 x}{\sin^2 x}dx=\int\frac{dx}{\sin^2 x}-x=-\cot x-x$　（ $\displaystyle\because\int\frac{dx}{\sin^2 x}=-\cot x$

295. $\displaystyle\int\sec xdx$　　（$z=\tan\dfrac{x}{2}$）

$$=\int\frac{dx}{\cos x}=\int\frac{2dz}{1-z^2}=\int\left(\frac{1}{1-z}+\frac{1}{1+z}\right)dz=\log\frac{1+z}{1-z}=\log\frac{1+\tan(x/2)}{1-\tan(x/2)}=\log\tan(\frac{\pi}{4}+\frac{x}{2})$$

$$= \log\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)} = \frac{1}{2}\log\{\ \frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}\ \}^2 = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}$$

$$= \frac{1}{2}\log\frac{(1+\sin x)^2}{1-\sin^2 x} = \frac{1}{2}\log(\frac{1+\sin x}{\cos x})^2 = \log\frac{1+\sin x}{\cos x} = \log(\sec x + \tan x)$$

$$= \log\frac{\cos x(1+\sin x)}{\cos^2 x} = \log\frac{\cos x}{1-\sin x} = -\log\frac{1-\sin x}{\cos x} = -\log(\sec x - \tan x)$$

296. $\int \sec^2 x dx = \int \frac{dx}{\cos^2 x} = \int (\tan x)' dx = \tan x$

298. $\int \cos ecx dx = \int \frac{dx}{\sin x}$ （$z = \tan\frac{x}{2}$）

$$= \int\frac{dz}{z} = \log\tan\frac{x}{2} = \frac{1}{2}\log\frac{\sin^2(x/2)}{\cos^2(x/2)} = \frac{1}{2}\log\frac{1-\cos x}{1+\cos x} = \frac{1}{2}\log\frac{(1-\cos x)^2}{1-\cos^2 x} = \log\frac{1-\cos x}{\sin x}$$

$$= \log(\cos ecx - \cot x) = \log\frac{1-\cos^2 x}{\sin x(1+\cos x)} = \log\frac{\sin x}{1+\cos x} = -\log\frac{1+\cos x}{\sin x} = -\log(\cos ecx + \cot x)$$

299. $\int \cos ec^2 x dx = \int \frac{dx}{\sin^2 x} = -\int (\frac{1}{\tan x})' dx = -\cot x$

301. $\int \frac{dx}{a+b\cos x}$ （$c^2 = \frac{b+a}{b-a}$ $z = \tan\frac{x}{2}$）

$$= \int\frac{\frac{2dz}{1+z^2}}{a+b\cdot\frac{1-z^2}{1+z^2}} = 2\int\frac{dz}{(a+b)+(a-b)z^2} = -\frac{2}{b-a}\int\frac{dz}{z^2-\frac{b+a}{b-a}} = -\frac{2}{b-a}\int\frac{dz}{z^2-c^2}$$

$$= \frac{1}{c(b-a)}(\int\frac{dz}{z+c} - \int\frac{dz}{z-c})$$

302. $\int \frac{dx}{a \pm b\sin x} = \int\frac{2dz}{a(1+z^2)\pm 2bz} = \int\frac{2dz}{a \pm 2bz + az^2}$ （$z = \tan\frac{x}{2}$）

303. $\int \frac{dx}{1+\sin x} = 2\int\frac{dz}{(z+1)^2} = -\frac{2}{z+1} = -\frac{2}{1+\tan(x/2)}$ （$z = \tan\frac{x}{2}$）

一方 $-\tan(\frac{\pi}{4}-\frac{x}{2}) = -\frac{1-\tan(x/2)}{1+\tan(x/2)} = \frac{\tan(x/2)-1}{1+\tan(x/2)}$ 従って $\int\frac{dx}{1+\sin x} = -\tan(\frac{\pi}{4}-\frac{x}{2})$

$\frac{\tan(x/2)-1}{1+\tan(x/2)} - (-\frac{2}{1+\tan(x/2)}) = 1$ （∵ 2 式の絶対値の差は 1、）

304. $\int \frac{dx}{1-\sin x} = 2\int\frac{dz}{(z-1)^2} = -\frac{2}{z-1} = -\frac{2}{\tan(x/2)-1} = \frac{2}{1-\tan(x/2)}$ （$z = \tan\frac{x}{2}$）

$\cot(\frac{\pi}{4}-\frac{x}{2}) = \frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}$ $\tan(\frac{\pi}{4}+\frac{x}{2}) = \frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}$

$\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)} - \frac{2}{1-\tan(x/2)} = -1$ で定数差、かつ $\cot(\frac{\pi}{4}-\frac{x}{2}) = \tan(\frac{\pi}{4}+\frac{x}{2})$

よって　$\int \frac{dx}{1-\sin x} = \cot(\frac{\pi}{4}-\frac{x}{2}) = \tan(\frac{\pi}{4}+\frac{x}{2})$

305. $\int \frac{dx}{1+\cos x} = 2\int \frac{dz}{(1+z^2)+(1-z^2)} = \int dz = z = \tan\frac{x}{2}$　　（$z=\tan\frac{x}{2}$）

また　$\cos ecx - \cot x = \frac{1-\cos x}{\sin x} = \frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)} = \tan\frac{x}{2}$　から成立する。

306. $\int \frac{dx}{1-\cos x} = 2\int \frac{dz}{(1+z^2)-(1-z^2)} = \int \frac{dz}{z^2} = -\frac{1}{z} = -\cot\frac{x}{2}$　　（$z=\tan\frac{x}{2}$）

また　$= -\cot x - \cos ecx = \frac{-1-\cos x}{\sin x} = \frac{-2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)} = -\cot\frac{x}{2}$　から成立する。

307. $\int \frac{dx}{a+b\sin x} = 2\int \frac{dz}{az^2+2bz+a} = \frac{2}{a}\int \frac{dz}{(z+\frac{b}{a})^2+\frac{a^2-b^2}{a^2}}$　　（$|x|<\pi$）　（$z=\tan\frac{x}{2}$）

(1)　$a^2>b^2$ のとき　$z+\frac{b}{a} = \frac{\sqrt{a^2-b^2}}{a}\tan\theta$　として　$\tan\theta = \frac{az+b}{\sqrt{a^2-b^2}}$

与式$= \frac{2}{a}\cdot\frac{a}{\sqrt{a^2-b^2}}\theta = \frac{2}{\sqrt{a^2-b^2}}\theta = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{a\tan(x/2)+b}{\sqrt{a^2-b^2}}$

(2)　$a^2<b^2$ のとき　部分分数分解して

与式$= \frac{2}{a}\left(\int \frac{dz}{z+\frac{b}{a}-\frac{\sqrt{b^2-a^2}}{a}} - \int \frac{dz}{z+\frac{b}{a}+\frac{\sqrt{b^2-a^2}}{a}}\right)$

$$= \frac{2}{a}\cdot\frac{a}{2\sqrt{b^2-a^2}}\log\frac{z+\frac{b}{a}-\frac{\sqrt{b^2-a^2}}{a}}{z+\frac{b}{a}+\frac{\sqrt{b^2-a^2}}{a}} = \frac{1}{\sqrt{b^2-a^2}}\log\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}$$

(3)　$\tanh^{-1}\frac{a\tan(x/2)+b}{\sqrt{b^2-a^2}} = u$　とおくと 定義から　$\tanh u = \frac{e^u-e^{-u}}{e^u+e^{-u}} = \frac{a\tan(x/2)+b}{\sqrt{b^2-a^2}}$

これから　$e^{2u} = \frac{a\tan(x/2)+b+\sqrt{b^2-a^2}}{a\tan(x/2)+b-\sqrt{b^2-a^2}}$　　$\therefore u = \frac{1}{2}\log\frac{a\tan(x/2)+b+\sqrt{b^2-a^2}}{a\tan(x/2)+b-\sqrt{b^2-a^2}}$

求める積分は$\frac{-2}{\sqrt{b^2-a^2}}u = \frac{1}{\sqrt{b^2-a^2}}\tan^{-1}\frac{a\tan(x/2)+b-\sqrt{b^2-a^2}}{a\tan(x/2)+b+\sqrt{b^2-a^2}}$　と等しいから

$\int \frac{dx}{a+b\sin x} = \frac{-2}{\sqrt{b^2-a^2}}\tanh^{-1}\frac{a\tan(x/2)+b}{\sqrt{b^2-a^2}}$　またこの式は$\coth^{-1}x = \tanh^{-1}x$　から

$\coth^{-1}$ の時も成り立つ。309.以降も頻出するが、これら 2 式について触れない。

308. $\int\frac{dx}{a+b\sin x}=\frac{1}{b\cos\alpha}\log\frac{\sin\frac{x+\alpha}{2}}{\cos\frac{x-\alpha}{2}}$　但し$a=b\sin\alpha$　$\sqrt{b^2-a^2}=b\cos\alpha$

与式$=\int\frac{\frac{2dz}{1+z^2}}{a+b\cdot\frac{2z}{1+z^2}}=\int\frac{\frac{2dz}{1+z^2}}{b(\sin\alpha+\frac{2z}{1+z^2})}=\frac{2}{b}\int\frac{dz}{z^2\sin\alpha+2z+\sin\alpha}$　$(|x|<\pi)(z=\tan\frac{x}{2})$

$$=\frac{2}{b\sin\alpha}\int\frac{dz}{(z+\frac{1}{\sin\alpha})^2-\frac{\cos^2\alpha}{\sin^2\alpha}}=\frac{1}{b\cos\alpha}(\int\frac{dz}{z+\frac{1-\cos\alpha}{\sin\alpha}}-\int\frac{dz}{z+\frac{1+\cos\alpha}{\sin\alpha}})$$

$$=\frac{1}{b\cos\alpha}\{log(\frac{\sin(x/2)}{\cos(x/2)}+\frac{\sin(\alpha/2)}{\cos(\alpha/2)})-\log(\frac{\sin(x/2)}{\cos(x/2)}+\frac{\cos(\alpha/2)}{\sin(\alpha/2)})\}$$

$$=\frac{1}{b\cos\alpha}\{log(\frac{\sin\frac{x}{2}\cos\frac{\alpha}{2}+\cos\frac{x}{2}\sin\frac{\alpha}{2}}{\cos\frac{x}{2}\cos\frac{\alpha}{2}})-\log(\frac{\sin\frac{x}{2}\sin\frac{\alpha}{2}+\cos\frac{x}{2}\cos\frac{\alpha}{2}}{\cos\frac{x}{2}\sin\frac{\alpha}{2}})\}$$

$$=\frac{1}{b\cos\alpha}(\log\sin\frac{x+\alpha}{2}-\log\cos\frac{x-\alpha}{2}-\log\cos\frac{x}{2}-\log\cos\frac{\alpha}{2}+\log\cos\frac{x}{2}+\log\sin\frac{\alpha}{2})$$

$=\frac{1}{b\cos\alpha}(\log\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}+\log\frac{\sin\frac{x+\alpha}{2}}{\cos\frac{x-\alpha}{2}})$　定数　$\frac{1}{b\cos\alpha}\log\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$　を除き、解を得る。

309. $\int\frac{dx}{a+b\cos x}=2\int\frac{dz}{(a-b)z^2+a+b}$　　$(|x|<\pi)$　$(z=\tan\frac{x}{2})$

(1)　$a^2-b^2>0$　のとき　$z=\sqrt{\frac{a+b}{a-b}}\tan\theta$ とすると　$dz=\sqrt{\frac{a+b}{a-b}}\cdot\frac{d\theta}{\cos^2\theta}$、$\tan\theta=\sqrt{\frac{a-b}{a+b}}z$

与式$=\frac{2}{a-b}\int\frac{dz}{z^2+\frac{a+b}{a-b}}=\frac{2}{\sqrt{a^2-b^2}}\theta=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{\sqrt{a^2-b^2}\tan(x/2)}{a+b}$

(2)　$a^2-b^2<0$　のとき　　与式$=\frac{2}{a-b}\int\frac{dz}{z^2-\frac{b+a}{b-a}}=-\frac{2}{b-a}\cdot\frac{\sqrt{b-a}}{2\sqrt{b+a}}\int(\frac{dz}{z-\sqrt{\frac{b+a}{b-a}}}$

$$-\int\frac{dz}{z+\sqrt{\frac{b+a}{b-a}}})=-\frac{1}{\sqrt{b^2-a^2}}\log\frac{z-\sqrt{\frac{b+a}{b-a}}}{z+\sqrt{\frac{b+a}{b-a}}}=\frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2}-\sqrt{b+a}}$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b^2 - a^2}\ \tan(x/2) + a + b}{\sqrt{b^2 - a^2}\ \tan(x/2) - a - b}$$

310. $\displaystyle\int \frac{dx}{a + b\tan x} = \frac{1}{a^2 + b^2}\left[b\log(a\cos x + b\sin x) + ax\right]$ （$z = \tan x$）

$$I = \int \frac{\frac{dz}{1+z^2}}{a + bz} = \int \frac{dz}{(a+bz)(z^2+1)}$$

k, l, m, n を未定係数として

$$\frac{lz + m}{z^2 + 1} - \frac{n}{bz + a} = \frac{(bl - n) + (al + bm)z + am - n}{(z^2+1)(bz+a)}$$

から $z^2, z,$ 定数項の各係数は $bl - n = 0 \quad al + bm = 0 \quad am - n = 1$

これを解いて

$$l = -\frac{b}{a^2 + b^2}, \quad m = \frac{a}{a^2 + b^2}, \quad n = -\frac{b^2}{a^2 + b^2}$$

$$I = \frac{1}{a^2+b^2}\left(\int \frac{b^2}{bz+a} - \int \frac{bz - a}{z^2+1}dz\right) = \frac{1}{a^2+b^2}\left(\int \frac{b(bz+a)'}{bz+a}dz - \int \frac{\frac{1}{2}b(z^2+1)' - a}{z^2+1}dz\right)$$

$$= \frac{1}{a^2+b^2}\left\{b\log(bz+a) - \frac{1}{2}b\log(z^2+1) + a\int \frac{dz}{z^2+1}\right\}$$

$$= \frac{1}{a^2+b^2}\left\{\frac{1}{2}b\log\frac{(bz+a)^2}{z^2+1} + a\tan^{-1} z\right\}$$

$$= \frac{1}{a^2+b^2}\left\{b\log(a + b\tan x)\cos x + ax\right\} = \frac{1}{a^2+b^2}\left\{b\log(a\cos x + b\sin x) + ax\right\}$$

311. $\displaystyle\int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2}\sin(x + \frac{\pi}{4})} = \frac{1}{\sqrt{2}}\int co\sec(x + \frac{\pi}{4})dx = \frac{1}{\sqrt{2}}\log\tan(\frac{x}{2} + \frac{\pi}{8})$

312. $\displaystyle\int \frac{\sin x}{a + b\cos x}dx = -\frac{1}{b}\int \frac{(a + b\cos x)'}{a + b\cos x}dx = -\frac{1}{b}\log(a + b\cos x)$

313. $\displaystyle\int \frac{(a' + b'\cos x)dx}{a + b\cos x} = \frac{b'}{b}x + \frac{a'b - ab'}{b}\int \frac{dx}{a + b\cos x}dx$

$\displaystyle\frac{b'}{b}x + \frac{a'b - ab'}{b} \cdot \frac{1}{b\cos x + a} = \frac{b'\cos x + a'}{b\cos x + a}$ ←割り算の結果から

2行目の式を （$b\cos x + a$） で割り積分すると証明すべき式が得られる

323. $\displaystyle\int \frac{dx}{a + b\sin^2 x}$ （$|x| < \frac{\pi}{2}$） （$z = \tan x$）

$\displaystyle dz = \frac{dx}{\cos^2 x} = (1 + \tan^2 x)dx$ から $\displaystyle dx = \frac{dz}{1 + z^2} \quad \sin^2 x = \frac{\sin^2 x}{\sin^2 x + \cos^2 x} = \frac{z^2}{1 + z^2}$

与式$=\int \frac{dz}{(a+b)z^2+a}=\frac{1}{a+b}\int \frac{dz}{z^2+\frac{a}{a+b}}$　　(1) 分母第2項 $a^2+ab>0$ 　のとき

$z=\sqrt{\frac{a}{a+b}}\tan\theta$ 　とし $dz=\sqrt{\frac{a}{a+b}}\cdot\frac{d\theta}{\cos^2\theta}$ 　　$\tan\theta=\sqrt{\frac{a+b}{a}}z$

$$=\frac{1}{a+b}\cdot\sqrt{\frac{a+b}{a}}\theta=\frac{1}{\sqrt{a^2+ab}}\tan^{-1}\sqrt{\frac{a+b}{a}}\tan x=\frac{1}{\sqrt{a^2+ab}}\tan^{-1}\frac{\sqrt{a^2+ab}\tan x}{a}$$

(2) 分母第2項 $a^2+ab<0$ のとき

与式$=\frac{1}{a+b}\int \frac{dz}{z^2-(\sqrt{\frac{-a}{a+b}})^2}=\frac{1}{a+b}\cdot\frac{\sqrt{a+b}}{2\sqrt{-a}}\int\left(\frac{1}{z-\sqrt{\frac{-a}{a+b}}}-\frac{1}{z+\sqrt{\frac{-a}{a+b}}}\right)dz$

$$=\frac{1}{2\sqrt{-a^2-ab}}\log\frac{\sqrt{a+b}z-\sqrt{-a}}{\sqrt{a+b}z+\sqrt{-a}}=\frac{1}{2\sqrt{-a^2-ab}}\log\frac{\sqrt{-a^2-ab}\tan x+a}{\sqrt{-a^2-ab}\tan x-a}$$

324. $\int \frac{dx}{a+b\cos^2 x}$　　　　($|x|<\frac{\pi}{2}$)　($z=\tan x$)

dz,dx は上記と同様　$\cos^2 x=\frac{\cos^2 x}{\sin^2 x+\cos^2 x}=\frac{1}{1+z^2}$　与式$=\int \frac{dz}{az^2+a+b}$

$=\frac{1}{a}\int \frac{dz}{z^2+(\sqrt{\frac{a+b}{a}})^2}$　　(1) $a^2+ab>0$時　$=\frac{1}{\sqrt{a^2+ab}}\tan^{-1}\sqrt{\frac{a}{a+b}}\tan x$

$=\frac{1}{\sqrt{a^2+ab}}\tan^{-1}\frac{\sqrt{a^2+ab}\tan x}{a+b}$　　　(2) $a^2+ab<0$ のとき

与式$=\frac{1}{a}\int \frac{dz}{z^2-(\sqrt{\frac{a+b}{-a}})^2}=\frac{1}{a}\cdot\frac{\sqrt{-a}}{2\sqrt{a+b}}\int\left(\frac{1}{z-\sqrt{\frac{a+b}{-a}}}-\frac{1}{z+\sqrt{\frac{a+b}{-a}}}\right)dz$

$$=-\frac{1}{2\sqrt{-a^2-ab}}\log\frac{\sqrt{-a}z-\sqrt{a+b}}{\sqrt{-a}z+\sqrt{a+b}}=\frac{1}{2\sqrt{-a^2-ab}}\log\frac{\sqrt{-a^2-ab}\tan x+a+b}{\sqrt{-a^2-ab}\tan x-a-b}$$

325. $\int \frac{dx}{a\cos^2 x+b\sin^2 x}=\int \frac{\frac{dx}{\cos^2 x}}{a+b\tan^2 x}$　　　($|x|<\frac{\pi}{2}$)　($z=\tan x$)

$dz=\frac{dx}{\cos^2 x}$　　(1) $ab>0$ のとき　与式$=\int \frac{dz}{a+bz^2}=\frac{1}{b}\int \frac{dz}{z^2+\frac{a}{b}}$

$z=\sqrt{\dfrac{a}{b}}\tan\theta$ とおくと　$dz=\sqrt{\dfrac{a}{b}}\cdot\dfrac{d\theta}{\cos^2\theta}$，$\tan\theta=\sqrt{\dfrac{b}{a}}z$　これから　$=\dfrac{1}{\sqrt{ab}}\theta$

$$=\frac{1}{\sqrt{ab}}\tan^{-1}\sqrt{\frac{b}{a}}z==\frac{1}{\sqrt{ab}}\tan^{-1}\frac{\sqrt{ab}\tan x}{a}$$

(2)　$ab<0$ のとき

$$\int\frac{dz}{z^2-(\sqrt{(-a/b)})^2}=\frac{1}{b}\cdot\frac{1}{2}\sqrt{\frac{b}{-a}}\int\left\{\frac{1}{z-\sqrt{(-a/b)}}-\frac{1}{z+\sqrt{(-a/b)}}\right\}dz=\frac{1}{2\sqrt{-ab}}\log\frac{\sqrt{b}z-\sqrt{-a}}{\sqrt{b}z+\sqrt{-a}}$$

$$=\frac{1}{2\sqrt{-ab}}\log\frac{\sqrt{-ab}z-\sqrt{(-a)^2}}{\sqrt{-ab}z+\sqrt{(-a)^2}}=\frac{1}{2\sqrt{-ab}}\log\frac{\sqrt{-ab}\tan x+a}{\sqrt{-ab}\tan x-a}$$

326. $\displaystyle\int\frac{\sin x\cos x dx}{a\cos^2 x+b\sin^2 x}=\frac{1}{2(b-a)}\int\frac{(a\cos^2 x+b\sin^2 x)'dx}{a\cos^2 x+b\sin^2 x}=\frac{1}{2(b-a)}\log(a\cos^2 x+b\sin^2 x)$

328. $\displaystyle\int\frac{dx}{a+b\cos x+c\sin x}=\frac{2}{\sqrt{a^2-b^2-c^2}}\tan^{-1}\frac{(a-b)\tan(x/2)+c}{\sqrt{a^2-b^2-c^2}}$　あるいは

$$=\frac{1}{\sqrt{b^2+c^2-a^2}}\log\frac{(a-b)\tan(x/2)+c-\sqrt{b^2+c^2-a^2}}{(a-b)\tan(x/2)+c+\sqrt{b^2+c^2-a^2}}$$　（$|x|<\pi$）（$z=\tan\dfrac{x}{2}$）

$$\int\frac{\dfrac{2dz}{1+z^2}}{a+b\cdot\dfrac{1-z^2}{1+z^2}+c\cdot\dfrac{2z}{1+z^2}}=\int\frac{2dz}{(a-b)z^2+2cz+a+b}=\int\frac{\dfrac{2}{a-b}dz}{(z+\dfrac{c}{a-b})^2+\dfrac{a^2-b^2-c^2}{(a-b)^2}}$$

(1)　$a^2>b^2+c^2$ のとき　$z+\dfrac{c}{a-b}=\dfrac{\sqrt{a^2-b^2-c^2}}{a-b}\tan\theta$　とおいて

$\tan\theta=\dfrac{a-b}{\sqrt{a^2-b^2-c^2}}z+\dfrac{c}{\sqrt{a^2-b^2-c^2}}$　　$dz=\dfrac{\sqrt{a^2-b^2-c^2}}{a-b}\cdot\dfrac{d\theta}{\cos^2\theta}$

与式$=\dfrac{2}{a-b}\cdot\dfrac{a-b}{\sqrt{a^2-b^2-c^2}}\theta=\dfrac{2}{\sqrt{a^2-b^2-c^2}}\tan^{-1}\dfrac{(a-b)\tan(x/2)+c}{\sqrt{a^2-b^2-c^2}}\theta$

(2)　$a^2<b^2+c^2$ のとき　同様に

与式$=\dfrac{1}{a-b}\cdot\dfrac{a-b}{\sqrt{b^2+c^2-a^2}}\log\dfrac{z+(c/(a-b))-(\sqrt{b^2+c^2-a^2}/(a-b))}{z+(c/(a-b))+(\sqrt{b^2+c^2-a^2}/(a-b))}$

$$=\frac{1}{\sqrt{b^2+c^2-a^2}}\log\frac{(a-b)\tan(x/2)+c-\sqrt{b^2+c^2-a^2}}{(a-b)\tan(x/2)+c+\sqrt{b^2+c^2-a^2}}$$

329. $\displaystyle\int\frac{dx}{a(1+\cos x)+c\sin x}=\int\frac{\frac{2dz}{1+z^2}}{a(1+\frac{1-z^2}{1+z^2})+c\cdot\frac{2z}{1+z^2}}=\int\frac{dz}{a+cz}$　　（$z=\tan\frac{x}{2}$ ）

$$=\frac{1}{c}\log(a+cz)=\frac{1}{c}\log(a+c\tan\frac{x}{2})$$

331. $\displaystyle\int\frac{(x+\sin x)dx}{1+\cos x}=\int\frac{x}{1+\cos x}dx+\int\frac{\sin x}{1+\cos x}dx=\int(\tan\frac{x}{2})'xdx-\int\frac{(1+\cos x)'}{1+\cos x}dx$

$$=x\tan\frac{x}{2}-\int\tan\frac{x}{2}dx-\log(1+\cos x)=x\tan\frac{x}{2}+2\int\frac{(\cos(x/2))'}{\cos(x/2)}dx-\log 2\cos^2\frac{x}{2}$$

$=x\tan\frac{x}{2}+2\log\cos\frac{x}{2}-2\log\cos\frac{x}{2}-\log 2$　定数 $-\log 2$ を除いて　与式＝$x\tan\frac{x}{2}$

338. $\displaystyle\int\frac{xdx}{1+\sin x}=\int x\cdot\left\{-\tan(\frac{\pi}{4}-\frac{x}{2})\right\}'dx=-x\tan(\frac{\pi}{4}-\frac{x}{2})+\int\tan(\frac{\pi}{4}-\frac{x}{2})dx$

$$\int\tan(\frac{\pi}{4}-\frac{x}{2})dx=\int\frac{1-\sin x}{\cos x}dx=\int(\sec x-\tan x)dx=\log\tan(\frac{\pi}{4}+\frac{x}{2})+\log\cos x$$

$$=\log\frac{\{(\cos(x/2)+\sin(x/2)\}\{\cos^2(x/2)-\sin^2(x/2)\}}{\cos(x/2)-\sin(x/2)}=2\log(\cos\frac{x}{2}+\sin\frac{x}{2})$$

$$=2\log\sqrt{2}\cos(\frac{\pi}{4}-\frac{x}{2})=\log 2+2\log\cos(\frac{\pi}{4}-\frac{x}{2})$$

これから定数 $\log 2$ を除き　$\displaystyle\int\frac{xdx}{1+\sin x}=-x\tan(\frac{\pi}{4}-\frac{x}{2})+2\log\cos(\frac{\pi}{4}-\frac{x}{2})$　を得る。

339. $\displaystyle\int\frac{xdx}{1-\sin x}=\int x\cdot\left\{cot(\frac{\pi}{4}-\frac{x}{2})\right\}'dx=x\cot(\frac{\pi}{4}-\frac{x}{2})-\int\cot(\frac{\pi}{4}-\frac{x}{2})dx$

ここで　$\displaystyle\int\cot(\frac{\pi}{4}-\frac{x}{2})dx=\int\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}dx=\int\frac{(\cos\frac{x}{2}+\sin\frac{x}{2})^2}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}dx=\int\frac{1+\sin x}{\cos x}dx$

$$=\log\tan(\frac{\pi}{4}+\frac{x}{2})-\log\cos x=\log\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{(\cos\frac{x}{2}-\sin\frac{x}{2})^2\cos x}=\log\frac{1}{1-\sin x}=-\log(\cos\frac{x}{2}-\sin\frac{x}{2})^2$$

$$=-2\log\sqrt{2}\sin(\frac{\pi}{4}-\frac{x}{2})=-\log 2-2\log\sin(\frac{\pi}{4}-\frac{x}{2})$$

定数 $\log 2$ を除き　与式＝$x\cot(\frac{\pi}{4}-\frac{x}{2})+2\log\sin(\frac{\pi}{4}-\frac{x}{2})$

340. $\displaystyle\int\frac{xdx}{1+\cos x}=\int x\cdot(\tan\frac{x}{2})'dx=x\tan\frac{x}{2}-\int\tan\frac{x}{2}dx=x\tan\frac{x}{2}+2\int\frac{(\cos(x/2))'}{\cos(x/2)}dx$

$$=x\tan\frac{x}{2}+2\log\cos\frac{x}{2}$$

341. $\int\frac{xdx}{1-\cos x}=\int x\cdot(-\cot\frac{x}{2})'\,dx=-x\cot\frac{x}{2}+\int\cot\frac{x}{2}dx=-x\cot\frac{x}{2}+2\int\frac{(\sin(x/2))'}{\sin(x/2)}dx$

$$=-x\cot\frac{x}{2}+2\log\sin\frac{x}{2}$$

343. $\int\frac{dx}{a+b\tan^2 x}$　$z=\tan x$ とおいて　与式$=\int\frac{dz}{(a+bz^2)(1+z^2)}$　（$z=\tan x$）

被積分関数を $\frac{pz+q}{a+bz^2}-\frac{rz+s}{1+z^2}$ とおいて、未定係数を求める。

$q-as=1$　$p-ar=0$　$q-bs=0$　$p-br=0$　これから　$p=r=0$

$q=\frac{b}{b-a}$　$s=\frac{1}{b-a}$　から　与式$=\frac{1}{b-a}\int(\frac{b}{a+bz^2}-\frac{1}{1+z^2})dz=\frac{1}{b-a}\int\frac{dz}{z^2+(a/b)}$

$-\frac{1}{b-a}\int\frac{dz}{1+z^2}$　第１項で　$z=\sqrt{\frac{a}{b}}\tan\theta$　$dz=\sqrt{\frac{a}{b}}\cdot\frac{d\theta}{\cos^2\theta}$　$\theta=\tan^{-1}\sqrt{\frac{b}{a}}\tan x$

$=\frac{1}{b-a}\cdot\sqrt{\frac{b}{a}}\tan^{-1}\sqrt{\frac{b}{a}}z-\frac{1}{b-a}\tan^{-1}z=\frac{1}{b-a}\{\sqrt{\frac{b}{a}}\tan^{-1}(\sqrt{\frac{b}{a}}\tan x)-\tan^{-1}\tan x\}$

$=\frac{1}{a-b}\{x-\sqrt{\frac{b}{a}}\tan^{-1}(\sqrt{\frac{b}{a}}\tan x\}$

344. $\int\frac{\tan xdx}{a+b\tan x}=\int\frac{zdz}{(a+bz)(1+z^2)}=\frac{1}{a^2+b^2}\left\{\int\frac{az+b}{1+z^2}dz-\int\frac{abdz}{a+bz}\right\}$　（$z=\tan x$）

$$=\frac{1}{a^2+b^2}\left\{\frac{1}{2}\int\frac{a(1+z^2)'}{1+z^2}dz+b\int\frac{dz}{1+z^2}-\int\frac{a(a+bz)'dz}{a+bz}\right\}$$

$$=\frac{a}{2(a^2+b^2)}\{-\log(a+bz)^2+\log(1+z^2)\}+\frac{b}{a^2+b^2}\tan^{-1}z$$

$$=-\frac{a}{2(a^2+b^2)}\log(a+b\tan x)^2\cos^2 x+\frac{b}{a^2+b^2}x=\frac{1}{a^2+b^2}\{bx-a\log(a+b\tan x)$$

$$-a\log\cos x\}=\frac{1}{a^2+b^2}\{bx-a\log(a+b\tan x)+a\log\sec x\}$$

345. $\int x\sin xdx=\int(-\cos x)'xdx=-x\cos x+\int\cos xdx=\sin x-x\cos x$

346. $\int x^2\sin xdx=\int(-\cos x)'x^2dx=-x^2\cos x+2\int x\cos xdx=-x^2\cos x$

$$+2(x\sin x-\int\sin xdx)=-x^2\cos x+2x\sin x+2\cos x=2x\sin x-(x^2-2)\cos x$$

347 $\int x^3\sin xdx=-x^3\cos x+3\int x^2\cos xdx=-x^3\cos x+3(x^2\sin x-2\int x\sin xdx)$

$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x = (3x^2-6)\sin x - (x^3-6x)\cos x$$

348. $\int x^m \sin x dx = \int (-\cos x)' x^m dx = -x^m \cos x + m\int x^{m-1}\cos x dx$

353. $\int \frac{\sin x}{x^m}dx = \int (\frac{x^{-m+1}}{-m+1})' \sin x dx = -\frac{1}{m-1}\cdot\frac{\sin x}{x^{m-1}} + \frac{1}{m-1}\int\frac{\cos x}{x^{m-1}}dx$

355. $\int \frac{x}{\sin^2 x}dx = \int x(-\cot x)' dx = -x\cot x - \int(-\cot x)dx = -x\cot x + \log\sin x$

356. $\int \frac{x}{\cos^2 x}dx = \int x(\tan x)' dx = x\tan x - \int \tan x dx = x\tan x + \log\cos x$

364. $\int \sin mx \sin nx dx = \frac{1}{2}\int \{\cos(m-n)x - \cos(m+n)x\}dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}$

365 $\int \sin mx \cos nx dx = \frac{1}{2}\int \{\sin(m+n)x + \sin(m-n)x\}dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)}$

366 $\int \cos mx \cos nx dx = \frac{1}{2}\int \{\cos(m+n)x + \cos(m-n)x\}dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)}$

369. $\int \sin(mx+a)\sin(nx+b)dx = -\frac{1}{2}\int [\cos\{(m+n)x+a+b\} - \cos\{(m-n)x+a-b\}]dx$

$$= -\frac{\sin\{(m+n)x+a+b]}{2(m+n)} + \frac{\sin\{(m-n)x+a-b\}}{2(m-n)}$$

370. $\int \cos(mx+a)\cos(nx+b)dx = \frac{1}{2}\int [\cos\{(m+n)x+a+b\} + \cos\{(m-n)x+a-b\}]dx$

$$= \frac{\sin\{(m+n)x+a+b]}{2(m+n)} + \frac{\sin\{(m-n)x+a-b\}}{2(m-n)}$$

371. $\int \sin(mx+a)\cos(nx+b)dx = \frac{1}{2}\int [\sin\{(m+n)x+a+b\} + \sin\{(m-n)x+a-b\}]dx$

$$= -\frac{\cos\{(m+n)x+a+b]}{2(m+n)} - \frac{\cos\{(m-n)x+a-b\}}{2(m-n)}$$

372. $\int \sin^2 mx dx = \frac{1}{2}\int(1-\cos 2mx)dx = \frac{1}{2}(x - \frac{1}{2m}\sin 2mx) = \frac{1}{2m}(mx - \sin mx\cos mx)$

373. $\int \cos^2 mx dx = \frac{1}{2}\int(1+\cos 2mx)dx = \frac{1}{2}(x + \frac{1}{2m}\sin 2mx) = \frac{1}{2m}(mx + \sin mx\cos mx)$

374. $\int \sin mx \cos mx dx = \frac{1}{2}\int \sin 2mx dx = -\frac{1}{4m}\cos 2mx$

375. $\int \sin nx \sin^m x dx = I(m,n)$ とおく。

$$I(m,n) = \int\left(-\frac{\cos nx}{n}\right)' \sin^m x dx = -\frac{1}{n}\cos nx \sin^m x + \frac{m}{n}\int\{\cos(n-1)x \cdot \sin^{m-1} x - \sin nx \sin^m x\}dx$$

$$= -\frac{1}{n}\cos nx \sin^m x + \frac{m}{n}\int \cos(n-1)x \cdot \sin^{m-1} x dx - \frac{m}{n} I(m,n)$$

これから　$\int \sin nx \sin^m x dx = \frac{1}{m+n}\left\{-\cos nx \sin^{m-1} x + m\int \cos(n-1)x \cdot \sin^{m-1} x dx\right\}$

378. $\int \cos nx \cos^m x dx = I(m,n)$ とおき 375. 同様変形し、$I(m,n)$ について解く。

$$I(m,n) = \int\left(\frac{\sin nx}{n}\right)' \cos^m x dx = \frac{1}{n}\sin nx \cos^m x + \frac{m}{n}\int \cos(n-1)x \cdot \cos^{m-1} x dx - \frac{m}{n} I(m,n)$$

386. $\int \sin ax \sin bx \sin cx dx = -\frac{1}{2}\int \cos(a+b)x \sin cx dx + \frac{1}{2}\int \cos(a-b)x \sin cx dx$

$$= \frac{1}{4}\int \{\sin(a-b+c)x + \sin(b+c-a)x + \sin(a+b-c)x - \sin(a+b+c)x\} dx$$

$$= -\frac{1}{4}\left[\frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c}\right]$$

387. $\int \cos ax \cos bx \cos cx dx = \frac{1}{2}\int \cos(a+b)x \cos cx dx + \frac{1}{2}\int \cos(a-b)x \cos cx dx$

$$= \frac{1}{4}\int \{\cos(a+b+c)x + \cos(a+b-c)x + \cos(a-b+c)x + \cos(b+c-a)x\} dx$$

$$= \frac{1}{4}\left[\frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(c+a-b)x}{c+a-b} + \frac{\sin(a+b-c)x}{a+b-c}\right]$$

388. $\int \sin ax \cos bx \cos cx dx = \frac{1}{2}\int \sin(a+b)x \cos cx dx + \frac{1}{2}\int \sin(a-b)x \cos cx dx$

$$= \frac{1}{4}\int \{\sin(a+b+c)x + \sin(a+b-c)x + \sin(a-b+c)x + \sin(a-b-c)x\} dx$$

$$= -\frac{1}{4}\left[\frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(c+a-b)x}{c+a-b}\right]$$

389. $\int \cos ax \sin bx \sin cx dx = \frac{1}{2}\int \sin(a+b)x \sin cx dx - \frac{1}{2}\int \sin(a-b)x \sin cx dx$

$$= \frac{1}{4}\int \{\cos(a+b-c)x + \cos(a-b+c)x - \cos(a+b+c)x - \cos(b+c-a)x\} dx$$

$$= \frac{1}{4}\left[\frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a-b+c)x}{a-b+c} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a}\right]$$

以　上